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Synchronal algorithm and cyclic algorithm for fixed point problems and variational inequality problems in hilbert spaces

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Abstract

We design synchronal algorithm and cyclic algorithm based on the general iterative algorithm proposed by Tian in 2010 for finding the common fixed point x^* of finite family of strict pseudo-contractive mappings which is the solution of the variational inequality $\langle(\gamma f - \mu F)x^*, x - x^*\rangle \leq 0$, $\forall x \in \bigcap_{i=1}^{N} F(T_i)$.

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1. Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $||\cdot||$, respectively. Let C be nonempty closed subset of H.

Recall that a mapping $T : C \to H$ is said to be k-strict pseudo-contraction if there exists a constant $k \in [0, 1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2 \quad \forall x, y \in C. \tag{1.1}$$

These mappings are extensions of nonexpansive mappings which satisfy the inequality (1.1) with $k = 0$. That is, $T : C \to H$ is nonexpansive if

$$||Tx - Ty|| \leq ||x - y||, \quad \forall x, y \in C.$$

We denote by $F(T)$ the set of fixed points of the mapping T, that is

$$F(T) = \{x \in H : Tx = x\}.$$

We assume that $F(T) \neq \emptyset$ it is well known that $F(T)$ is closed convex.

Let $F : C \to H$ be a nonlinear operator, we consider the problem of finding a point $x^* \in C$ such that

$$\langle Fx^*, x - x^*\rangle \geq 0 \quad \forall x \in C.$$

We denote by $V I(F, C)$ the set of solutions of this variational inequality problem.

Takahashi [1] introduced a classical CQ algorithm as follows:

$$\begin{cases} x_0 \in C \text{ is arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C: \ ||y_n - z|| \ \le \ ||x_n - z||\}, \\ Q_n = \{z \in C: \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \ n = 0, 1, 2 \ldots \end{cases}$$

where T is nonexpansive mapping, and $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$. Then they showed that $\{x_n\}$ converged strongly to $P_{F(T)}(x_0)$ by the hybrid method in the mathematical programming. But it is hard to compute by this algorithm, because projection has to be used in every process.

The hybrid steepest descent method of Yamada [2] conquered this deficiency and proposed the following algorithm for solving the variational inequality.

Take $x_0 \in H$ arbitrarily and define $\{x_n\}$ by

$$x_{n+1} = Tx_n - \mu\lambda_n F(Tx_n). \tag{1.2}$$

where T is a nonexpansive mapping on H, F is L-Lipschitzian and η-strongly monotone with $k > 0$, $\eta > 0$, $0 < \mu < 2\eta/L^2$. If $\{\lambda_n\}$ is a sequence in (0, 1) satisfying the following conditions:

(i) $\lim_{n\to\infty} \lambda_n = 0$;

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$

(iii) either $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ or $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$,

then the sequence $\{x_n\}$ converged strongly to the unique solution of the variational inequality

$$\langle F\tilde{x}, x - \tilde{x}\rangle \ge 0 \quad \forall x \in F(T).$$

Besides, he also proposed cyclic algorithm:

$$x_{n+1} = T^{\lambda_n} x_n = (I - \mu\lambda_n F)T_{[n]}x_n,$$

where $T_{[n]} = T_{n \bmod N}$, he also got strong convergence theorems.

On the other hand, Marino and Xu [3] considered the following general iterative method: an initial x_0 is selected in H arbitrarily

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \tag{1.3}$$

where T is a nonexpansive mapping on H, f is a contraction, A is a linear bounded strongly positive operator, and $\{\alpha_n\}$ is a sequence in (0, 1) satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) either $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

They proved that the sequence $\{x_n\}$ converged strongly to a fixed point $\tilde{x}$ of T which solves the variational inequality

$$\langle(\gamma f - A)\tilde{x}, x - \tilde{x}\rangle \leq 0 \quad \forall x \in F(T).$$

Very recently, Tian [4] combined the iterative method (1.3) with the Yamada's method (1.2) and considered the following general iterative method

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n, \tag{1.4}$$

where T is a nonexpansive mapping on H, f is a contraction, and F is k- Lipschitzian and η-strongly monotone with $k > 0$, $\eta > 0$, $0 < \mu < 2\eta/k^2$.

He proved that if the sequence $\{\alpha_n\}$ of parameters satisfies (C1)-(C3), then the sequence $\{x_n\}$ generated by (1.4) converged strongly to a fixed point $\tilde{x}$ of T which solves the variational inequality

$$\langle(\gamma f - \mu F)\tilde{x}, x - \tilde{x}\rangle \leq 0 \quad \forall x \in F(T). \tag{1.5}$$

In this paper we designed two algorithms for finding a common fixed point x^* of finite strict pseudo-contractions which also solves the variational inequality

$$\langle(\gamma f - \mu G)x^*, x - x^*\rangle \leq 0 \quad \forall x \in \bigcap_{i=1}^{N} F(T_i), \tag{1.6}$$

where $N \geq 1$ is a positive integer and $\{T_i\}_{i=1}^N$ are N strict pseudo-contractions.

Let T be defined by

$$T = \sum_{i=1}^{N} \lambda_i T_i,$$

Where $\lambda_i > 0$ such that $\sum_{i=1}^N \lambda_i = 1$. We will show that the sequence $\{x_n\}$ generated by the algorithm:

$$\begin{cases} T^{\beta_n} = \beta_n I + (1 - \beta_n)\sum_{i=1}^N \lambda_i T_i, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)T^{\beta_n} x_n \end{cases} \tag{1.7}$$

will converge strongly to a solution to the problem (1.6).

Another approach to the problem (1.6) is the cyclic algorithm. For each $i = 1, \ldots, N$, let

$$A_i = \beta_i I + (1 - \beta_i)T_i,$$

where the constant β_i satisfies $k_i < \beta_i < 1$. Beginning with $x_0 \in H$, we define the sequence $\{x_n\}$ cyclically by

$$\begin{aligned} x_1 &= \alpha_0 \gamma f(x_0) + (I - \alpha_0 \mu G)(A_1 x_0), \\ x_2 &= \alpha_1 \gamma f(x_1) + (I - \alpha_1 \mu G)(A_2 x_1), \\ &\vdots \\ x_N &= \alpha_{N-1} \gamma f(x_{N-1}) + (I - \alpha_{N-1} \mu G)(A_N x_{N-1}), \\ x_{N+1} &= \alpha_N \gamma f(x_N) + (I - \alpha_N \mu G)(A_1 x_N), \\ &\vdots \end{aligned}$$

Indeed, the algorithm above can be written as

$$\begin{cases} A_{[n]} = \beta_{[n]}I + (1-\beta_{[n]})T_{[n]}, \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I-\alpha_n\mu G)A_{[n+1]}x_n, \end{cases} \tag{1.8}$$

where $T_{[n]} = T_i$, with $i = n(\mathrm{mod} N)$, $1 \le i \le N$. We will show that this cyclic algorithm (1.8) is also strongly convergent if the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriately chosen.

We will use the notations:

1. $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
2. $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak !-limit set of $\{x_n\}$.

2. Preliminaries

We need some facts and tools which are listed as below.

Definition 1 A mapping $F : C \to H$ is called η-strongly monotone if there exists a positive constant $\eta > 0$ such that

$$\langle Fx - Fy, x - y\rangle \ge \eta\|x-y\|^2, \quad \forall x, y \in C.$$

Definition 2 B is called to be strongly positive bounded linear operator on H, if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Bx, x\rangle \ge \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

Lemma 2.1. *(see* [5]*) Let C be a nonempty closed convex subset of a real Hilbert space H and T : C ! C is a nonexpansive mapping. If a sequence* $\{x_n\}$ *in C such that* $x_n \rightharpoonup z$ *and* $(I - T)x_n \to 0$, *then* $z = Tz$.

Lemma 2.2. *(see* [6]*) Let C be a nonempty closed convex subset of a real Hilbert space H. If* $T : C \to C$ *is a* κ*-strict pseudo-contraction, then the mapping I - T is demiclosed at 0. That is, if* $\{x_n\}$ *is a sequence in C such that* $x_n \rightharpoonup \tilde{x}$*and* $(I - T)x_n \to 0$, *then* $(I-T)\tilde{x} = 0$.

Lemma 2.3. *(see* [7]*) Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \quad n \ge 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that:*

(i) $\lim_{n\to\infty}\gamma_n = 0$ *and* $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(ii) $\lim_{n\to\infty}\delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. *(see* [4]*) Let H be a real Hilbert space,* $f : H \to H$ *a contraction with coefficient* $0 < \alpha < 1$, *and* $F : H \to H$ *a k-Lipschitzian continuous operator and* η*-strongly monotone operator with* $k > 0$, $\eta > 0$. *Then for* $0 < \gamma < \mu\eta/\alpha$,

$$\langle x-y, (\mu F - \gamma f)x - (\mu F - \gamma f)y\rangle \ge (\mu\eta - \gamma\alpha)\|x-y\|^2, \quad \forall x, y \in H.$$

That is, $\mu F - \gamma f$ *is strongly monotone with coefficient* $\mu\eta - \gamma\alpha$.

Lemma 2.5. *(see* [8]*) Suppose* $S : C \to H$ *is a k-strict pseudo-contraction. Define* $T : C \to H$ *by* $Tx = \lambda x + (1 - \lambda)Sx$ *for each* $x \in C$. *Then, as* $\lambda \in [k, 1)$, *T is a nonexpansive mapping such that* $F(T) = F(S)$.

Lemma 2.6. *(see* [6]*) Assume C is a closed convex subset of a Hilbert space H. Given an integer $N \geq 1$, assume for each $1 \leq i \leq N$, $T_i : C \to C$ is a k_i-strict pseudo-contraction for some $0 \leq k_i < 1$. Assume $\{\gamma_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \gamma_i = 1$. Suppose that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ then*

$$F(T) = \bigcap_{i=1}^{N} F(T_i).$$

Lemma 2.7. *(see* [9]*) Assume $T_i : H \to H$ is a k_i-strict pseudo-contraction for some $0 \leq k_i < 1$ $(1 \leq i \leq N)$: Let $T_{\alpha_i} = \alpha_i I + (1 - \alpha_i)T_i$, $k_i < \alpha_i < 1$ $(1 \leq i \leq N)$, if $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, then*

$$F(T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_N}) = \bigcap_{i=1}^{N} F(T_{\alpha_i}).$$

Lemma 2.8. *Let $F : H \to H$ be a η-strongly monotone and L-Lipschitzian operator with $L > 0$, $\eta > 0$. Assume that $0 < \mu < 2\eta/L^2$, $\tau = \mu(\eta - \frac{\mu L^2}{2})$ and $0 < t < 1$. Then $||(I - \mu tF)x - (I - \mu tF)y|| \leq (1 - t\tau)\, ||x - y||$.*

Proof. Put $g = I\ \mu tF$, then

$$\begin{aligned} ||gx - gy||^2 &= \langle gx - gy, gx - gy \rangle \\ &= \langle x - y - \mu t(Fx - Fy), x - y - \mu t(Fx - Fy) \rangle \\ &= ||x - y||^2 - 2\mu t \langle x - y, Fx - Fy \rangle + \mu^2 t^2 ||Fx - Fy||^2 \\ &\leq ||x - y||^2 - 2\mu t\eta ||x - y||^2 + \mu^2 t^2 L^2 ||x - y||^2 \\ &= (1 - 2\mu t\eta + \mu^2 t^2 L^2)||x - y||^2. \end{aligned}$$

Therefore,

$$\begin{aligned} ||gx - gy|| &\leq \sqrt{1 - 2\mu t(\eta - \frac{\mu L^2}{2})}||x - y|| \\ &\leq (1 - \tau t)||x - y||, \end{aligned}$$

that is,

$$||(I - \mu tF)x - (I - \mu tF)y|| \leq (1 - t\tau)||x - y||.$$

□

3. Synchronal algorithm

Theorem 3.1. *Let H be a real Hilbert space and let $T_i : H \to H$ be a k_i-strict pseudo-contraction for some $k_i \in (0, 1)$ $(i = 1,..., N)$ such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, f be a contraction with coefficient $\beta \in (0, 1)$ and λ_i be a positive constant such that $\sum_{i=1}^N \lambda_i = 1$. Let $G : H \to H$ be a η-strongly monotone and L-Lipschitzian operator with $L > 0$, $\eta > 0$. Assume that $0 < \mu < 2\eta/L^2$, $0 < \gamma < \mu(\eta - \frac{\mu L^2}{2})/\beta = \tau/\beta$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in (0, 1), satisfying the following conditions:*

(3.1a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;;

(3.1b) $\sum_{n=1}^{\infty} | \alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} | \beta_{n+1} - \beta_n| < \infty$;
(3.1c) $0 \le \max_i k_i \le \beta_n < a < 1$ *for all* $n \ge 0$;
let $\{x_n\}$ *be the sequences define d by the composite process (1.7), i.e.*

$$\begin{cases} T^{\beta_n} = \beta_n I + (1-\beta_n)\sum_{i=1}^{N} \lambda_i T_i, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G) T^{\beta_n} x_n \end{cases}$$

Then $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_i\}_{i=1}^{N}$*which solves the variational inequality (1.6).*

Proof. Put $T = \sum_{i=1}^{N} \lambda_i T_i$, then by Lemma 2.6, we conclude that T is a k-strict pseudo-contraction with $k = \max \{k_i : 1 \le i \le N\}$ and $F(T) = \bigcap_{i=1}^{N} F(T_i)$.

We can rewrite the algorithm (1.7) as

$$\begin{cases} T^{\beta_n} = \beta_n I + (1-\beta_n)T, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G) T^{\beta_n} x_n \end{cases}$$

Furthermore, by Lemma 2.5, we conclude that T^{β_n} is a nonexpansive mapping and $F(T^{\beta_n}) = F(T)$.

Step 1. $\{x_n\}$ is bounded.

Take $v \in \bigcap_{i=1}^{N} F(T_i)$, from (1.7) and Lemma 2.9 we have

$$\begin{aligned} ||x_{n+1} - v|| &= ||\alpha_n \gamma f(x_n) + (I - \alpha_n \mu G) T_{\beta_n} x_n - v|| \\ &= ||\alpha_n(\gamma f(x_n) - \mu G v) + (I - \alpha_n \mu G) T_{\beta_n} x_n - (I - \alpha_n \mu G) v|| \\ &\le (1 - \alpha_n \tau)||x_n - v|| + \alpha_n ||\gamma(f(x_n) - f(v)) + \gamma f(v) - \mu G v|| \\ &\le (1 - \alpha_n \tau)||x_n - v|| + \alpha_n \gamma \beta ||x_n - v|| + \alpha_n ||\gamma f(v) - \mu G v|| \\ &= (1 - \alpha_n(\tau - \gamma\beta))||x_n - v|| + \alpha_n ||\gamma f(v) - \mu G v|| \\ &\le \max\left\{||x_n - v||, \frac{||\gamma f(v) - \mu G v||}{\tau - \gamma\beta}\right\}. \end{aligned}$$

By simple induction, we have

$$||x_n - v|| \le \max\left\{||x_0 - v||, \frac{||\gamma f(v) - \mu G v||}{\tau - \gamma\beta}\right\}.$$

Hence $\{x_n\}$ is bounded.

From $v \in \bigcap_{i=1}^{N} F(T_i)$, we have $v \in F(T)$, hence

$$\begin{aligned} ||Tx_n - v||^2 &\le ||x_n - v||^2 + k||(I - T)x_n - (I - T)v||^2 \\ &= ||x_n - v||^2 + k||(x_n - Tx_n)||^2 \\ &= ||x_n - v||^2 + k||(x_n - v) + (v - Tx_n)||^2 \\ &= (1 + k)||x_n - v||^2 + k||Tx_n - v||^2 + 2k\langle x_n - v, v - Tx_n\rangle \\ &\le (1 + k)||x_n - v||^2 + k||Tx_n - v||^2 + 2k||x_n - v||\,||Tx_n - v||. \end{aligned}$$

It follows that

$$(1 - k)||Tx_n - v||^2 - 2k||x_n - v||\,||Tx_n - v|| - (1 + k)||x_n - v||^2 \le 0.$$

So, we have

$$||Tx_n - v|| \leq \frac{1+k}{1-k}||x_n - v||.$$

Therefore, $\{Tx_n\}$ is bounded.

G is L-Lipschitzian, so

$$||GTx_n - GTv|| \leq L||Tx_n - Tv||$$

$\{Tx_n\}$ is bounded, so $\{GT^{\beta_n}x_n\}$ is bounded.

f is a contraction, so $f(x_n)$ is bounded.

Step 2.

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.1}$$

Observing that

$$\begin{cases} x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)T^{\beta_n}x_n, \\ x_n = \alpha_{n-1}\gamma f(x_{n-1}) + (I - \alpha_{n-1}\mu G)T^{\beta_{n-1}}x_{n-1}, \end{cases}$$

we have

$$\begin{aligned} x_{n+1} - x_n = {} & \alpha_n\gamma(f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1})\gamma f(x_{n-1}) \\ & + [(I - \alpha_n\mu G)T^{\beta_n}x_n - (I - \alpha_n\mu G)T^{\beta_{n-1}}x_{n-1}] \\ & + (\alpha_{n-1} - \alpha_n)\mu GT^{\beta_{n-1}}x_{n-1}. \end{aligned}$$

This in turn implies that

$$\begin{aligned} ||x_{n+1} - x_n|| & \leq \alpha_n\gamma\beta||x_n - x_{n-1}|| + (1 - \alpha_n\tau)||T^{\beta_n}x_n - T^{\beta_{n-1}}x_{n-1}|| \\ & \quad + |\alpha_n - \alpha_{n-1}|(\gamma||f(x_{n-1})|| + \mu||GT^{\beta_{n-1}}x_{n-1}||) \\ & \leq \alpha_n\gamma\beta||x_n - x_{n-1}|| + (1 - \alpha_n\tau)||T^{\beta_n}x_n - T^{\beta_{n-1}}x_{n-1}|| \\ & \quad + |\alpha_n - \alpha_{n-1}|M_1, \end{aligned} \tag{3.2}$$

where M_1 is an appropriate constant such that $M_1 \geq \sup_{n\geq 1}\{\gamma||f(x_n)|| + \mu||GT^{\beta_n}x_n||\}$. On the other hand, we note that

$$\begin{aligned} |T^{\beta_n}x_n - T^{\beta_{n-1}}x_{n-1}|| & \leq ||T^{\beta_n}x_n - T^{\beta_n}x_{n-1}|| + ||T^{\beta_n}x_{n-1} - T^{\beta_{n-1}}x_{n-1}|| \\ & \leq ||\beta_n x_{n-1} + (1-\beta_n)Tx_{n-1} - \beta_{n-1}x_{n-1} - (1-\beta_{n-1})Tx_{n-1}|| \\ & \quad + ||x_n - x_{n-1}|| \\ & \leq ||x_n - x_{n-1}|| + |\beta_n - \beta_{n-1}|\ ||x_{n-1} - Tx_{n-1}|| \\ & \leq ||x_n - x_{n-1}|| + |\beta_n - \beta_{n-1}|M_2, \end{aligned} \tag{3.3}$$

where M_2 is an appropriate constant such that $M_2 \geq \sup_{n\geq 1} \{||x_n - Tx_n||\}$. Substituting (3.3) into (3.2) yields

$$\begin{aligned} ||x_{n+1} - x_n|| & \leq \alpha_n\gamma\beta||x_n - x_{n-1}|| + (1 - \alpha_n\tau)||x_n - x_{n-1}|| + |\alpha_n - \alpha_{n-1}|M_1 \\ & \quad + |\beta_n - \beta_{n-1}|M_2 \\ & \leq (1 - \alpha_n(\tau - \gamma\beta))||x_n - x_{n-1}|| + M_3(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|), \end{aligned}$$

where M_3 is an appropriate constant such that $M_3 \geq \max\{M_1, M_2\}$. By conditions (3.1a) and (3.1b) and Lemma 2.3, we obtain that $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$.

From (1.7), we observe that

$$\begin{aligned}||x_{n+1} - T^{\beta_n}x_n|| &= \alpha_n||\gamma f(x_n) + \mu GT^{\beta_n}x_n|| \\ &\le \alpha_n(||f(x_n) - f(v)|| + ||f(v) + GT^{\beta_n}v|| + ||GT^{\beta_n}x_n - GT^{\beta_n}v||).\end{aligned}$$

It follows from the condition (3.1a) and the boundedness of $\{f(x_n)\}$ and $\{GT^{\beta_n}x_n\}$ that

$$\lim_{n\to\infty} ||x_{n+1} - T^{\beta_n}x_n|| = 0. \tag{3.4}$$

On the other hand,

$$\begin{aligned}||x_{n+1} - T^{\beta_n}x_n|| &= ||x_{n+1} - [\beta_n x_n + (1-\beta_n)Tx_n]|| \\ &= ||(x_{n+1} - x_n) + (1-\beta_n)(x_n - Tx_n)|| \\ &\ge (1-\beta_n)||x_n - Tx_n|| - ||(x_{n+1} - x_n)||.\end{aligned}$$

Hence, by condition (3.1c), we have

$$\begin{aligned}||x_n - Tx_n|| &\le \frac{1}{1-\beta_n}[||x_{n+1} - T^{\beta_n}x_n|| + ||(x_{n+1} - x_n)||] \\ &\le \frac{1}{1-a}[||x_{n+1} - T^{\beta_n}x_n|| + ||(x_{n+1} - x_n)||].\end{aligned}$$

From (3.1) and (3.4), we obtain

$$\lim_{n\to\infty} ||x_n - Tx_n|| = 0. \tag{3.5}$$

From the boundedness of $\{x_n\}$, we deduced that $\{x_n\}$ converges weakly. Assume $x_n \rightharpoonup p$, by Lemma 2.2 and (3.5), we obtain $p = Tp$. So, we have

$$\omega_w(x_n) \subset F(T). \tag{3.6}$$

Notice by Lemma 2.4, μG - γf is strongly monotone, so the variational inequality (1.6) has a unique solution $x^* \in F(T)$.

Step 3.

$$\limsup_{n\to\infty}\langle(\gamma f - \mu G)x^*, x_n - x^*\rangle \le 0. \tag{3.7}$$

Indeed, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - \mu G)x^*, x_n - x^*\rangle = \lim_{j\to\infty}\langle(\gamma f - \mu G)x^*, x_{n_j} - x^*\rangle.$$

Without loss of generality, we may further assume that $x_{n_j} \rightharpoonup x$. It follows from (3.6) that $x \in F(T)$. Since x^* is the unique solution of (1.6), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma f - \mu G)x^*, x_n - x^*\rangle &= \lim_{j\to\infty}\langle(\gamma f - \mu G)x^*, x_{n_j} - x^*\rangle \\ &= \langle(\gamma f - \mu G)x^*, x - x^*\rangle \le 0.\end{aligned}$$

Step 4.

$$\lim_{n\to\infty} ||x_n - x^*|| = 0. \tag{3.8}$$

From Lemma 2.9, we have

$$\begin{aligned}
|x_{n+1} - x^*||^2 &= ||\alpha_n(\gamma f(x_n) - \mu Gx^*) + (I - \alpha_n\mu G)T^{\beta_n}x_n - (I - \alpha_n\mu G)x^*||^2 \\
&\le (1 - \alpha_n\tau)^2||x_n - x^*||^2 + 2\alpha_n\langle \gamma f(x_n) - \mu Gx^*, x_{n+1} - x^*\rangle \\
&\le (1 - \alpha_n\tau)^2||x_n - x^*||^2 + 2\alpha_n\gamma\langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - \mu Gx^*, x_{n+1} - x^*\rangle \\
&\le (1 - \alpha_n\tau)^2||x_n - x^*||^2 + 2\alpha_n\gamma\beta||x_n - x^*||\ ||x_{n+1} - x^*|| \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - \mu Gx^*, x_{n+1} - x^*\rangle \\
&\le (1 - \alpha_n\tau)^2||x_n - x^*||^2 + \alpha_n\gamma\beta(||x_n - x^*||^2 + ||x_{n+1} - x^*||^2) \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - \mu Gx^*, x_{n+1} - x^*\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\le \frac{1 - 2\alpha_n\tau + (\alpha_n\tau)^2 + \alpha_n\gamma\beta}{1 - \alpha_n\gamma\beta}||x_n - x^*||^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n\gamma\beta}\langle \gamma f(x^*) - \mu Gx^*, x_{n+1} - x^*\rangle \\
&\le [1 - \frac{2(\tau - \gamma\beta)\alpha_n}{1 - \alpha_n\gamma\beta}]||x_n - x^*||^2 + \frac{(\alpha_n\tau)^2}{1 - \alpha_n\gamma\beta}||x_n - x^*||^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n\gamma\beta}\langle \gamma f(x^*) - \mu Gx^*, x_{n+1} - x^*\rangle \\
&\le (1 - \gamma_n)||x_n - x^*||^2 + \delta_n,
\end{aligned}$$

where $\gamma_n := \frac{2\alpha_n(\tau - \gamma\beta)}{1 - \alpha_n\gamma\beta}$ and $\delta_n := \frac{\alpha_n}{1 - \alpha_n\gamma\beta}[\alpha_n\tau^2||x_n - x^*||^2 + 2\langle \gamma f(x^*) - \mu Gx^*, x_{n+1} - x^*\rangle]$.. $\gamma_n \le \frac{2(\tau - \gamma\beta)}{1 - \gamma\beta}\alpha_n$, from (3.1a), we have $\lim_{n\to\infty} \gamma_n = 0$; $\gamma_n \ge 2\alpha_n (\tau - \gamma\beta)$, from (3.1a), we have $\sum_{n=1}^{\infty} \gamma_n = \infty$; put $M = \sup \{||x_n - x^*|| : n \in N\}$, we have $\delta_n/\gamma_n = \frac{1}{2(\tau - \gamma\beta)}[\alpha_n\tau^2 M + 2\langle \gamma f(x^*) - \mu Gx^*, x_{n+1} - x^*\rangle]$. So, $\lim_{n\to\infty} \delta_n/\gamma_n \le 0$. Hence, by Lemma 2.3, we conclude that $x_n \to x^*$ as $n \to \infty$. □

4. Cyclic algorithm

Theorem 4.1. *Let H be a real Hilbert space and let $T_i : H \to H$ be a k_i-strict pseudo-contraction for some $k_i \in (0, 1)$ $(i = 1,..., N)$ such that $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$and f be a contraction with coefficient $\beta \in (0, 1)$. Let $G : H \to H$ be a η-strongly monotone and L-Lipschitzian operator with $L > 0$, $\eta > 0$. Assume that $0 < \gamma < \mu(\eta - \frac{\mu L^2}{2})/\beta = \tau/\beta$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in (0, 1), satisfying the following conditions:*

(4.1a) $\lim_{n\to\infty} \alpha_n = 0$,

(4.1b) $\sum_{n=0}^{\infty} \alpha_n = \infty$;;

(4.1c) $\sum_{n=1}^{\infty} |\ \alpha_{n+1} - \alpha_n| < \infty$, *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+N}} = 1$;

(4.1d) $\beta_{[n]} \in [k, 1)$, *where* $k = \max_i \{k_i : 1 \le i \le N\}$,

let $\{x_n\}$ *be the sequences define d by the composite process (1.8), i.e.*

$$\begin{cases} A_{[n]} = \beta_{[n]}I + (1 - \beta_{[n]})T_{[n]} \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n\mu G)A_{[n+1]}x_n, \end{cases}$$

where $T_{[n]} = T_i$, with $i = n(\mathrm{mod}N)$, $1 \leq i \leq N$, namely, $T_{[n]}$ is one of $T_1, T_2, \ldots, T_N$ circularly. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$ which solves the variational inequality (1.6).

Proof. **Step 1**. $\{x_n\}$ is bounded. Take $v \in \bigcap_{i=1}^N F(T_i)$, from (1.8) and Lemma 2.9 we have

$$\begin{aligned}
||x_{n+1} - v|| &= ||\alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)A_{[n+1]}x_n - v|| \\
&= ||\alpha_n(\gamma f(x_n) - \mu Gv) + (I - \alpha_n \mu G)A_{[n+1]}x_n - (I - \alpha_n \mu G)v|| \\
&\leq (1 - \alpha_n \tau)||x_n - v|| + \alpha_n||\gamma(f(x_n) - f(v)) + \gamma f(v) - \mu Gv|| \\
&\leq (1 - \alpha_n \tau)||x_n - v|| + \alpha_n \gamma \beta ||x_n - v|| + \alpha_n ||\gamma f(v) - \mu Gv|| \\
&= (1 - \alpha_n(\tau - \gamma\beta))||x_n - v|| + \alpha_n||\gamma f(v) - \mu Gv|| \\
&\leq \max\left\{||x_n - v||, \frac{||\gamma f(v) - \mu Gv||}{\tau - \gamma\beta}\right\}.
\end{aligned}$$

By simple induction, we have

$$||x_n - v|| \leq \max\left\{||x_0 - v||, \frac{||\gamma f(v) - \mu Gv||}{\tau - \gamma\beta}\right\}.$$

Hence $\{x_n\}$ is bounded.

From the proof of Step 1 in Section 3, we know that $\{T_{[n]}x_n\}$, $\{f(x_n)\}$, $\{GA_{[n]}x_n\}$ are bounded.

$$\begin{aligned}
||A_{[n]}x_n - A_{[n]}v|| &= ||(\beta_{[n]}I + (1 - \beta_{[n]})T_{[n]})x_n - (\beta_{[n]}I + (1 - \beta_{[n]})T_{[n]})v|| \\
&= ||\beta_{[n]}(x_n - v) + (1 - \beta_{[n]})(T_{[n]}x_n - v)|| \\
&\leq ||x_n - v|| + ||T_{[n]}x_n - v||.
\end{aligned}$$

So, $\{A_{[n]}x_n\}$ is bounded.

Step 2. $\lim_{n\to\infty} ||x_{n+N} - x_n|| = 0$.

By (1.8) and Lemma 2.9, we have

$$\begin{aligned}
||x_{n+N+1} - x_{n+1}|| &= ||\alpha_{n+N}\gamma f(x_{n+N}) + (I - \alpha_{n+N}\mu G)A_{[n+N+1]}x_{n+N} \\
&\quad - \alpha_n \gamma f(x_n) - (I - \alpha_n \mu G)A_{[n+1]}x_n|| \\
&= ||\alpha_{n+N}\gamma f(x_{n+N}) + (I - \alpha_{n+N}\mu G)A_{[n+1]}x_{n+N} \\
&\quad - \alpha_n \gamma f(x_n) - (I - \alpha_n \mu G)A_{[n+1]}x_n|| \\
&= ||\alpha_{n+N}\gamma f(x_{n+N}) - \alpha_{n+N}\gamma f(x_n) + \alpha_{n+N}\gamma f(x_n) \\
&\quad - \alpha_n \gamma f(x_n) + (I - \alpha_{n+N}\mu G)A_{[n+1]}x_{n+N} \\
&\quad - (I - \alpha_{n+N}\mu G)A_{[n+1]}x_n - (I - \alpha_n \mu G)A_{[n+1]}x_n|| \\
&\quad + (I - \alpha_{n+N}\mu G)A_{[n+1]}x_n \\
&\leq \alpha_{n+N}\gamma\beta||x_{n+N} - x_n|| + |\alpha_{n+N} - \alpha_n|\gamma||f(x_n)|| \\
&\quad + (1 - \alpha_n \tau)||x_{n+N} - x_n|| + |\alpha_{n+N} - \alpha_n|\mu||GA_{[n+1]}x_n|| \\
&\leq \alpha_{n+N}\gamma\beta||x_{n+N} - x_n|| + |\alpha_{n+N} - \alpha_n|K_1 \\
&\quad + (1 - \alpha_n \tau)||x_{n+N} - x_n|| \\
&= (1 - \alpha_n(\tau - \gamma\beta))||x_{n+N} - x_n|| + |\alpha_{n+N} - \alpha_n|K_1,
\end{aligned} \tag{4.1}$$

where K_1 is an appropriate constant such that $K_1 \geq \sup_{n\geq 1} \{\mu||GA_{[n+1]}x_n|| + \gamma||f(x_n)||\}$. By conditions (4.1a), (4.1b), (4.1c) and Lemma 2.3, we obtain $||x_{n+N} - x_n|| \to 0$ as $n \to \infty$.

Step 3. $\lim_{n\to\infty} ||x_n - A_{[n+N]} \cdots A_{[n+1]}x_n|| = 0$.

From (1.8), we observe that

$$||x_{n+1} - A_{[n+1]}x_n|| \ = \alpha_n||\gamma f(x_n) + \mu G A_{[n+1]}x_n||.$$

It follows from the condition (4.1a) and the boundedness of $\{f(x_n)\}$ and $\{GA_{[n+1]}x_n\}$ that

$$||x_{n+1} - A_{[n+1]}x_n|| \ \to 0(n \to \infty).$$

Recursively,

$$\begin{aligned} ||x_{n+N} - A_{[n+N]}x_{n+N-1}|| \ &\to 0(n \to \infty), \\ ||x_{n+N-1} - A_{[n+N-1]}x_{n+N-2}|| \ &\to 0(n \to \infty). \end{aligned}$$

By condition (4.1d) and Lemma 2.5, we know that $T^{\beta_{[n+N]},[n+N]}$ is nonexpansive, so we get

$$||A_{[n+N]}x_{n+N-1} - A_{[n+N]}A_{[n+N-1]}x_{n+N-2}|| \ \to 0(n \to \infty).$$

Proceeded accordingly, we have

$$\begin{aligned} ||A_{[n+N]}A_{[n+N-1]}x_{n+N-2} - A_{[n+N]}A_{[n+N-1]}A_{n+N-2}x_{n+N-3}|| \ &\to 0(n \to \infty), \\ &\vdots \\ ||A_{[n+N]} \cdots A_{[n+2]}x_{n+1} - A_{[n+N]} \cdots A_{[n+1]}x_n|| \ &\to 0(n \to \infty). \end{aligned}$$

Note that

$$\begin{aligned} |x_{n+N} - A_{[n+N]} \cdots A_{[n+1]}x_n|| \ \le\ & ||x_{n+N} - A_{[n+N]}x_{n+N-1}|| \\ & + ||A_{[n+N]}x_{n+N-1} - A_{[n+N]}A_{[n+N-1]}x_{n+N-2}|| \\ & + \cdots \\ & + ||A_{[n+N]} \cdots A_{[n+2]}x_{n+1} - A_{[n+N]} \cdots A_{[n+1]}x_n|| \end{aligned}$$

From all the expressions above, we obtain

$$||x_{n+N} - A_{[n+N]} \cdots A_{[n+1]}x_n|| \ \to 0(n \to \infty).$$

Since

$$||x_n - A_{[n+N]} \cdots A_{[n+1]}x_n|| \ \le \ ||x_n - x_{n+N}|| + ||x_{n+N} - A_{[n+N]} \cdots A_{[n+1]}x_n||,$$

we conclude $||x_n - A_{[n+N]} \cdots A_{[n+1]}x_n|| \to 0(n \to \infty)$.

Step 4.

$$\omega_w(x_n) \subset \bigcap_{i=1}^{N} F(T_i). \tag{4.2}$$

Take a subsequence $\{x_{n_j}\} \subset \{x_n\}$, by step 3, we get

$$||x_{n_j} - A_{[n_j+N]} \cdots A_{[n_j+1]}x_{n_j}|| \ \to 0(j \to \infty).$$

Notice that, for each n_j, $A_{[n_j+N]}A_{[n_j+N-1]} \cdots A_{[n_j+1]}$ is some permutation of the mappings $A_1A_2 \cdots A_N$, since $A_1, A_2, \cdots, A_N$ are finite, all the finite permutation are $N!$, there must be some permutation appears infinite times.

Without loss of generality, suppose this permutation is $A_1 A_2 \cdots A_N$, we can take a subsequence $\{x_{n_{j_k}}\} \subset \{x_{n_j}\}$ such that $x_{n_{j_k}} \rightharpoonup q(k \to \infty)$ and

$$||x_{n_{j_k}} - A_1A_2 \cdots A_N x_{n_{j_k}}|| \to 0(k \to \infty).$$

By Lemma 2.5, we conclude that A_1, A_2,···, A_N are all nonexpansive. It is easy to prove that $A_{[n_j+N]} \cdots A_{[n_j+1]}$ is nonexpansive, so A_1 A_2···A_N is.

By Lemma 2.2, we have $q = A_1A_2 \cdots A_N q$. From Lemmas 2.5 and 2.7, we obtain

$$q \in F(A_1A_2 \cdots A_N) = \bigcap_{i=1}^{N} F(A_i) = \bigcap_{i=1}^{N} F(T_i).$$

Step 5.

$$\limsup_{n\to\infty} \langle (\gamma f - \mu F)x^*, x_n - x^* \rangle \leq 0. \tag{4.3}$$

Indeed, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (\gamma f - \mu G)x^*, x_n - x^* \rangle = \lim_{j\to\infty} \langle (\gamma f - \mu G)x^*, x_{n_j} - x^* \rangle.$$

Without loss of generality, we may further assume that $x_{n_j} \rightharpoonup x$. It follows from (4.2) that $x \in F(T)$. Since x^* is the unique solution of (1.6), we obtain

$$\begin{aligned}\limsup_{n\to\infty} \langle (\gamma f - \mu G)x^*, x_n - x^* \rangle &= \lim_{j\to\infty} \langle (\gamma f - \mu G)x^*, x_{n_j} - x^* \rangle \\ &= \langle (\gamma f - \mu G)x^*, x - x^* \rangle \leq 0.\end{aligned}$$

Step 6. $x_n \to x^*(n \to \infty)$.

From Lemma 2.9, we have

$$\begin{aligned}||x_{n+1} - x^*||^2 = {} & ||\alpha_n(\gamma f(x_n) - \mu G x^*) + (I - \alpha_n \mu G)A_{[n+1]}x_n - (I - \alpha_n \mu G)x^*||^2 \\ \leq {} & (1 - \alpha_n \tau)^2 ||x_n - x^*||^2 + 2\alpha_n \langle \gamma f(x_n) - \mu G x^*, x_{n+1} - x^* \rangle \\ \leq {} & (1 - \alpha_n \tau)^2 ||x_n - x^*||^2 + 2\alpha_n \gamma \langle f(x_n) - f(x^*), x_{n+1} - x^* \rangle \\ & + 2\alpha_n \langle \gamma f(x^*) - \mu G x^*, x_{n+1} - x^* \rangle \\ \leq {} & (1 - \alpha_n \tau)^2 ||x_n - x^*||^2 + 2\alpha_n \gamma \beta ||x_n - x^*|| \, ||x_{n+1} - x^*|| \\ & + 2\alpha_n \langle \gamma f(x^*) - \mu G x^*, x_{n+1} - x^* \rangle \\ \leq {} & (1 - \alpha_n \tau)^2 ||x_n - x^*||^2 + \alpha_n \gamma \beta (||x_n - x^*||^2 + ||x_{n+1} - x^*||^2) \\ & + 2\alpha_n \langle \gamma f(x^*) - \mu G x^*, x_{n+1} - x^* \rangle.\end{aligned}$$

This implies that

$$\begin{aligned}||x_{n+1} - x^*||^2 \leq {} & \frac{1 - 2\alpha_n \tau + (\alpha_n \tau)^2 + \alpha_n \gamma \beta}{1 - \alpha_n \gamma \beta} ||x_n - x^*||^2 \\ & + \frac{2\alpha_n}{1 - \alpha_n \gamma \beta} \langle \gamma f(x^*) - \mu G x^*, x_{n+1} - x^* \rangle \\ \leq {} & [1 - \frac{2(\tau - \gamma\beta)\alpha_n}{1 - \alpha_n \gamma \beta}] ||x_n - x^*||^2 + \frac{(\alpha_n \tau)^2}{1 - \alpha_n \gamma \beta} ||x_n - x^*||^2 \\ & + \frac{2\alpha_n}{1 - \alpha_n \gamma \beta} \langle \gamma f(x^*) - \mu G x^*, x_{n+1} - x^* \rangle \\ \leq {} & (1 - \gamma_n) ||x_n - x^*||^2 + \delta_n,\end{aligned}$$

where $\gamma_n := \frac{2\alpha_n(\tau - \gamma\beta)}{1 - \alpha_n \gamma \beta}$ and $\delta_n := \frac{\alpha_n}{1 - \alpha_n \gamma \beta} [\alpha_n \tau^2 ||x_n - x^*||^2 + 2\langle \gamma f(x^*) - \mu G x^*, x_{n+1} - x^* \rangle]$.. $\gamma_n \leq \frac{2(\tau - \gamma\beta)}{1 - \gamma\beta} \alpha_n$, from (4.1a), we have $\lim_{n\to\infty} \gamma_n = 0$; $\gamma_n \geq 2\alpha_n\ (\tau - \gamma\beta)$, from (4.1b), we have $\sum_{n=1}^{\infty}\gamma_n = \infty$; put $M = \sup\{||x_n - x^*||: n \in N\}$, we have $\delta_n/\gamma_n = \frac{1}{2(\tau-\gamma\beta)}[\alpha_n\tau^2 M + 2\langle\gamma f(x^*) - \mu Gx^*, x_{n+1} - x^*\rangle]$. So, $\limsup_{n\to\infty}\delta_n/\gamma_n \le 0$. Hence, by Lemma 2.3, we conclude that $x_n \to x^*$ as $n \to \infty$. . □

Taking $n = 1$, $\beta_n = 0$ and T is nonexpansive mapping in Theorems 3.1 and 4.1, we get

Corollary 1 (see[4]) Let $\{x_n\}$ be generated by the following algorithm

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \mu\alpha_n F)Tx_n,$$

Assume the sequence $\{\alpha_n\}$ satisfies conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$;
(C2) $\sum_{n=0}^{\infty}\alpha_n = \infty$;;
(C3) either $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n} = 1$

then $\{x_n\}$ converged strongly to $\tilde{x}$ which solves the variational inequality

$$\langle(\gamma f - \mu F)\tilde{x}, x - \tilde{x}\rangle \le 0, \quad \forall x \in F(T).$$

Taking $n = 1$, $\beta_n = 0$ and T is nonexpansive mapping, $G = A$, $\mu = 1$ in Theorems 3.1 and 4.1, we get

Corollary 2 (see[3]) Let $\{x_n\}$ be generated by the following algorithm:

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)Tx_n,$$

Assume the sequence $\{\alpha_n\}$ satisfies conditions (C1)-(C3), then the sequence $\{x_n\}$ converged strongly to a fixed point $\tilde{x}$ of T which solves the variational inequality

$$\langle(\gamma f - A)\tilde{x}, x - \tilde{x}\rangle \le 0 \quad \forall x \in F(T).$$

Taking $n = 1$, $\beta_n = 0$ and T is nonexpansive mapping, $\gamma = 0$ in Theorem 3.1 and Theorem 4.1, we get:

Corollary 3 (see[2]) Let $\{x_n\}$ be generated by the following algorithm

$$x_{n+1} = Tx_n - \mu\lambda_n F(Tx_n),$$

where T is a nonexpansive mapping on H, F is L-Lipschitzian and η-strongly monotone with $k > 0$, $\eta > 0$, $0 < \mu < 2\eta/L^2$. If $\{\lambda_n\}$ is a sequence in (0, 1) satisfies the following conditions:

(i) $\lim_{n\to\infty}\lambda_n = 0$;
(ii) $\sum_{n=0}^{\infty}\lambda_n = \infty$;
(iii) either $\sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$ or $\lim_{n\to\infty}\frac{\lambda_{n+1}}{\lambda_n} = 1$

then the sequence $\{x_n\}$ converged strongly to the unique solution of the variational inequality

$$\langle F\tilde{x}, x - \tilde{x}\rangle \ge 0, \quad \forall x \in F(T).$$

Taking $n = 1$, $\beta_n = 0$ and T is nonexpansive mapping, $\gamma = 0$ in Theorem 4.1, we get

Corollary 4 (see[2]) Let $\{x_n\}$ be generated by the following algorithm

$$x_{n+1} = T^{\lambda_n} x_n = (I - \mu\lambda_n F)T_{[n]}x_n$$

where $T_{[n]} = T_{n \bmod N}$. Assume $\{\lambda_n\}$ satisfies conditions (C1)-(C3) and $C = F(T_N \cdots T1) = F(T_1 T_N \cdots T_3 T_2) = \cdots = F(T_{N-1} T_{N-2} \cdots T_1 T_N)$, then $\{x_n\}$ converged strongly to the unique solution $\tilde{x} \in C$ of the variational inequality

$$\langle F\tilde{x}, x - \tilde{x}\rangle \geq 0 \quad \forall x \in C.$$

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All the authors read and approved the final manuscript.

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The authors declare that they have no completing interests.

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